When Not to Persevere

Nuances Related to Perseverance in Mathematical Problem Solving

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Abstract

While perseverance in mathematical problem solving has been of interest for some time, this construct is receiving renewed attention at present due in large part to its emphasis in the Mathematical Practices of the Common Core State Standards. In this paper, I consider nuances related to perseverance, particularly when persevering might be advantageous as well as when it might be inadvisable. In particular, I discuss the role of prior knowledge and metacognitive skills, the fine line between perseverance and stubbornness, and the importance of perceived task utility, in determining when perseverance is desirable in problem solving.
The notion of perseverance appears to have captured the interest and imagination of the mathematics education community at present. The Common Core State Standards (CCSS) explicitly mentions perseverance as integral to the first Standard for Mathematical Practice, in that students should “make sense of problems and persevere in solving them” (CCSS, 2010, p. 6). Furthermore, the recently released policy document from the National Council of Teachers of Mathematics (NCTM), *Principles to Actions*, notes that an effective teacher “provides students with appropriate challenges, encourages perseverance in solving problems, and supports productive struggle in learning mathematics” (NCTM, 2014, p. 11). A web search for sites related to perseverance in mathematics yields a plethora of lesson plans, videos, and blogs trumpeting the importance of perseverance and offering advice for how to support its development. Empirical support for the importance of perseverance is attributed to the results from the most recent Programme for International Student Assessment (PISA), which suggests that among students in higher-scoring countries, perseverance was strongly linked to higher performance on the 2012 assessment (OECD, 2013).

In this paper, my aim is to offer reflections on the topic of perseverance, by considering the conditions under which perseverance may be productive or even counter productive to successful problem solving. I seek to add complexity and nuance to the perception that perseverance is unquestionably an admirable or even a necessary attribute of mathematical problem solving. My hope is that my reflections spark further interest and research into the topic of perseverance.

**Defining Perseverance**

What is perseverance? The CCSS elaborates on the above-mentioned Standard for Mathematical Practice (“Make sense of problems and persevere in solving them,” CCSS, 2010, p. 6) by noting that mathematically proficient students engage with a problem by “looking for entry points to its solution,” which can lead them to “analyze givens, constraints, relationships, and goals,” to “make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt,” and to “monitor and evaluate their progress and change course if necessary” (p. 6). CCSS further elaborates that proficient students “check their answers to problems using a different method, and they continually ask themselves, ‘Does this make sense?’” (p. 6).

Perseverance is not explicitly defined in this paragraph. However, through a close reading of the words and examples used to describe this practice, we can make important observations about what the authors of the CCSS may have meant by perseverance. First, note that this
Standard for Mathematical Practice is worded as a compound phrase, with “make sense of problems” identified as distinct from “persevere in solving them.” This implies that making sense of problems is viewed as a sub-practice that is distinct from (although perhaps closely related to) persevering. Further evidence for the distinction between making sense and persevering can be found within the examples provided within this paragraph, in that some clearly refer to making sense (e.g., asking whether the answer makes sense) while others do not and, as a result, relate to perseverance (e.g., monitoring and evaluating progress).

Based on this one-paragraph description of the Practice, one interpretation of the authors’ intended message is that proficient students not only make sense of problems, but also make sense of the strategies used in trying to solve the problem—including considering which strategies can or should be used, and monitoring whether the application of these strategies results in desired outcomes. Perseverance would appear to relate to sense making of the problem-solving process, in that individuals should monitor the current state of the solution pathway and deliberately persist or change course as deemed necessary in order to continue progressing toward a solution. As such, CCSS appears to paint a picture of perseverance that is strongly metacognitive. Metacognition is frequently defined as the ability to assess the state of one’s own knowledge. Within a problem-solving context, metacognition refers to the ways that solvers plan, monitor, and assess their progress on a given problem. Decisions about persisting with a given strategy, changing to a new approach, or knowing if one is making progress on a problem all draw upon metacognition.

NCTM’s Principles to Actions also does not provide an explicit definition of perseverance, but it is similarly possible to make inferences about what the authors may have meant by this term. In particular, the vision of perseverance from Principles to Actions has more of a motivational flavor than the CCSS Standard for Mathematical Practice described above. Throughout Principles to Actions, sentences that use the word perseverance also include words such as challenge, difficulty, grapple, frustration, and productive. These word choices suggest that the authors view perseverance as something that one may be motivated to do when one faces difficulties in problem solving. In order to persevere, one needs to view the struggle that may inevitably be a part of problem solving as an opportunity to learn. Motivation enables a solver to see struggle as a natural part of the learning process, and to see that confronting and working through struggle can ultimately be helpful.

Further evidence for this view of the strong connections between perseverance and motivation within Principles to Actions can be found in its frequent use of the seemingly ubiquitous phrase productive struggle. Productive struggle is explicitly identified as a core
component of effective mathematics teaching and learning, implying that good teaching involves putting students in problem-solving situations where they will experience difficulties and frustrations that could serve as learning opportunities. Judging from the research that is cited in making points about productive struggle, the authors of *Principles to Actions* draw upon two currently popular literatures for supporting the link between productive struggle and perseverance.

First, links are made to research in educational psychology that relates to students’ beliefs about the nature of mathematical ability. Students who believe that they do not possess natural ability in mathematics and that math acumen is fixed and unchangeable may not be willing to persevere, while those who have a “growth mindset” (e.g., Blackwell, Trzesniewski, & Dweck, 2007; Dweck, 2006; see also Schoenfeld, 1989), or those who believe that with effort math abilities can continue to grow and develop, can persist and positively respond to problem-solving challenges. The idea that struggle can be productive appears to require a growth mindset and is tightly linked to perseverance.

Second, recent research by Kapur is often used to provide empirical support for the value of perseverance. Kapur frames his work around a construct that he calls *productive failure*. In several studies, Kapur found that students who initially experienced failure when attempting to tackle ill-structured problem-solving activities without the aid of instructional supports subsequently learned more when provided with instructional guidance (Kapur, 2008, 2009, 2010, 2012; Kapur & Bielaczyc, 2012). In Kapur’s studies, he deliberately places students in situations where they are likely to experience initial failure, in order to show that initial failure helps students to subsequently learn from instruction. The goal of Kapur’s work is not to see whether and why students persevere in the face of failure, but rather whether failing can, under certain conditions, be an effective precondition for learning. Kapur claims that when students persist in problem solving despite their failures (i.e., when they persevere), they can develop preparatory knowledge about the task on which they failed that subsequently makes them ready to learn from future structured instructional opportunities (e.g., see also Schwartz & Martin, 2004). Note that Kapur’s work does not indicate that students who persevere learn more than those who do not persevere, as his studies do not include a condition where students do not persevere. Rather, this work proposes that initial struggles can allow students to be more productive future learners—that prolonged, sustained, yet unsuccessful problem solving (i.e., perseverance) may be required to transform mere failure into *productive failure*.

Closely related to the motivational conception of perseverance that can be inferred from NCTM’s *Principles to Actions*, perseverance also makes an appearance in the most recent PISA
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2012 results. PISA explicitly considers perseverance to be a motivational variable. In student surveys administered by PISA, perseverance was operationalized from responses to four survey items: “When confronted with a problem, I give up easily,” “I put off difficult problems,” “I remain interested in the tasks that I start,” and “I continue working on tasks until everything is perfect” (OECD, 2013, p. 58). As with many surveys, the PISA team likely did not set out to create items that measured perseverance a priori, but rather (in some sort of factor analytic process) collected these four items that appeared to be thematically related and that performed similarly and post hoc grouped them together under the label of perseverance. In fact, not all of these four items seem to capture the motivational conception of perseverance that is implied in Principles to Actions. The first item is clearly the closest match, in that it indicates whether students do or do not give up easily when problem solving. But the second item seems to relate to procrastination, the third item is about interest, and the fourth item is about the desire to be complete and thorough while working on problems—none of which seem core to perseverance.

Within the domain of advanced mathematics, one could argue that perseverance has long been valued and perhaps even indispensible, especially given the importance placed on identifying and endeavoring to solve previously unsolved problems. For example, in an essay posted by the Mathematical Association of America, a mathematician comments,

> Ask any scientist or mathematician what it takes to make true, significant progress with a research question, and most often the answer is tenacity, patience, perseverance, deep care and consistency of thought, the confidence to learn from false leads (which means one must follow leads, even the false ones!), and plugging on day after day, week after week, month after month. One does this until a story of some kind emerges, even if that story goes against preconceived notions! (Tanton, n.d.)

Similarly, the German mathematician David Hilbert famously compiled a list of 23 unsolved mathematics problems in 1900—the majority of which are now generally believed to have been solved. More recently, the Clay Mathematics Institute posed seven unsolved problems in 2000 and offered a substantial cash prize for their solution. Mathematicians might not consider it unusual to spend years or even decades working on a single problem, which (even with such extreme perseverance) may not ultimately be solved or even be solvable. As a noteworthy recent example, Andrew Wiles famously took over six years to solve Fermat’s Last Theorem, which had (despite the efforts of mathematicians around the world for the past 350 years) remained unsolved.

Considering all of the above, there is clearly a great deal of overlap between conceptions of perseverance found in CCSS, NCTM, PISA, and from the discipline of mathematics. We can envision perseverance by considering a prototypical situation in which it occurs: While working
on a mathematics problem, a student does not face quick or immediate success but rather encounters challenges, difficulties, and possibly frustration. But she persists and does not give up, with the hope that continuing efforts will ultimately result in problem solving success. Persisting in the face of difficulties and challenge is perseverance. Perseverance is in some ways the opposite of insight, where a solution comes in a flash or instantly. To persevere is to toil away on a problem, perhaps for days or weeks; by virtue of persevering, one can ultimately arrive at the solution. Perseverance may be very difficult and perhaps not at all enjoyable, but through perseverance there is the hope that one can arrive at a problem solution.

When Is Perseverance Useful?

Armed with a clearer conception of what perseverance means, I now consider the questions that are central to this paper. When are students capable of persevering? When is it useful, and when is it not useful, to persevere? While the policy documents discussed above seem to identify perseverance as a universal good and a goal for all students, might there be circumstances where a student should not persevere? To explore these questions, I begin with three anecdotes that describe circumstances where perseverance may be neither productive nor desirable. I then step back from these anecdotes to identify three general principles that relate to when perseverance may be merited.

First, consider Benny, the title character in Erlwanger’s (1973) seminal case study that is considered part of the mathematics education research canon by most contemporary scholars. Benny was a 12-year-old boy with a history of difficulties in mathematics. Benny had participated in a program of individualized self-paced learning since he was in the second grade, and Erlwanger interviewed Benny several times during his 6th grade year to learn more about his mathematical knowledge related to decimals and fractions. Despite the fact that Benny had satisfactorily completed many units within the self-study curriculum over the years, Erlwanger quickly discovered that Benny’s understanding of rational number concepts and procedures was deeply flawed. In his attempt to move forward in the program and achieve mastery on assessments of progress, Benny appeared to have invented a variety of mathematically invalid rules that he consistently used and was able to explain and defend. While use of these rules enabled Benny to move quickly ahead in the curriculum, and while he did perform well on the assessments, it is also the case that Benny had developed deep and troubling misunderstandings about mathematics that would certainly inhibit his future progress in later courses.

Arguably, Benny exhibited considerable perseverance in his work within his math class. Although he had a history of math difficulties and was challenged by the independent self-paced
nature of the curriculum he used, he persisted. When he encountered problems that he did not know how to approach, he relied upon his ingenuity and tenacity by inventing rules that appeared to result in the correct answers. In hindsight, Benny would have been much better served had he essentially given up—acknowledged his inability to make sense of the mathematics, sought help from his teacher or his peers, and resisted the temptation to manufacture nonsensical mathematical rules that allowed him to continue making progress through the curriculum. In terms of his long-term success in mathematics, it seems safe to conclude that Benny’s perseverance did not serve him well.

A second and somewhat similar anecdote comes from one of the studies from my dissertation (Star, 2001). In several one-on-one tutoring sessions on the topic of linear equation solving, I worked closely with a 7th grade student to whom I refer as Kathy, who was enrolled in a pre-algebra class. I discovered that Kathy had developed an unusual and generally inefficient strategy for correctly solving linear equations that she consistently used. For example, given the equation \(2(x + 5) + 8(x - 1) = 7(x + 2)\), Kathy began by using the distributive property to ‘expand’ each of the distributed terms (see Figure 1). She then ‘moves’ each term from the right side to the left side, one at a time, until the left side is set equal to zero. She then moves constant terms back from the left to the right side, one at a time. When only variable terms remain on the left side, Kathy combines like variable terms, two terms at a time. Finally, she divides by the coefficient of the remaining variable term. In repeated tutoring sessions and on a variety of equations, Kathy regularly used this strategy to arrive at the correct solution.

\[
\begin{align*}
2(x + 5) + 8(x - 1) &= 7(x + 2) \\
2x + 10 + 8x - 8 &= 7x + 14 \\
2x + 10 + 8x - 8 - 14 &= 7x \\
2x + 10 + 8x - 8 - 14 - 7x &= 0 \\
2x + 10 + 8x - 14 - 7x &= 8 \\
2x + 10 + 8x - 7x &= 22 \\
2x + 8x - 7x &= 12 \\
10x - 7x &= 12 \\
3x &= 12 \\
x &= 4
\end{align*}
\]

*Figure 1.* Kathy’s strategy for solving a linear equation (adapted from Table 5.7 in Star, 2011, p. 80).

Although Kathy’s strategy enabled her to consistently arrive at the correct answer on a wide range of linear equations, it is not optimal. Her strategy is inefficient, particularly because it
includes redundant steps (e.g., the 14 was moved from the right to the left, only to be subsequently moved back from the left to the right). Furthermore, her persistent use of this strategy likely masked or perhaps compensated for deficiencies in her understanding about concepts underlying equation solving such as equivalence, like terms, and properties such as commutativity and associativity. Kathy’s strategy clearly needs optimization, but she was reluctant to deviate from this strategy. She continued to use it whenever she encountered a linear equation to solve, despite the future detriment that will likely result from her inflexibility and the deficiencies in her knowledge. Although she consistently was pressed by her teacher and me to problematize her strategy, Kathy persisted in its use—because she understood how to execute it and because it consistently yielded the correct answer. Refusal to deviate from an established and seemingly successful problem solving approach, even in the face of evidence that it may not be optimal in either the short or the long term, is another illustration of when perseverance does not seem productive.

Finally, my third anecdote comes from Greek mythology and the story of Sisyphus. As punishment for his hubris and a number of deceitful acts, Sisyphus was sentenced for all eternity to roll a huge rock up a steep hill. Yet whenever the rock approached the top of the hill, it would roll back down to the bottom, requiring Sisyphus to start again the process of pushing the rock up the hill. Consistent with this story, we use the word Sisyphean to describe tasks that can never be completed and/or are hopeless or futile.

Although Sisyphus did not have a choice in whether to continue to push the rock up the hill, this myth nevertheless points to another ill-advised illustration of perseverance. For Sisyphus, there is no hope that his persistence will ultimately yield success. Although his efforts may appear to be productive in the short term, it is absolutely certain that, regardless of how much he perseveres, he will always fail. Given the futility of his task, we can safely assume that if he had the choice, Sisyphus would decide not to persevere and give up—and that we would support his decision to disengage as a wise allocation of his efforts.

These three anecdotes are intended to illustrate circumstances when perseverance was not necessarily productive to problem solving. While perseverance generally seems a worthwhile goal, these anecdotes point to situations when it may in fact be more useful to give up rather than persist. Abstracting from these anecdotes, under what conditions is perseverance productive? Here I propose three general principles that can offer guidance about the potential utility of perseverance.

First, it seems clear to me that students’ prior knowledge should factor into decisions about when to persevere. In particular, **perseverance is only useful when students have**
sufficient prior knowledge and metacognitive abilities to make their struggles potentially productive. With respect to prior knowledge, if a student has little or no relevant prior knowledge that can be used to begin making progress on a problem, it is unclear whether she should persevere. For a solver who is unable to even make a first step toward solving a problem, persisting by staring at the blank page may be futile; the metaphor that describes the impossibility of pulling one’s self up by one’s bootstraps seems apt in such a situation. But perhaps even more importantly, perseverance is only potentially productive when solvers are able to monitor their own progress. Solvers need to know enough, and be able to view the state of their knowledge metacognitively, such that they can make a judgment about whether their efforts may lead to success with a reasonable probability. Core to perseverance is a nuanced metacognitive self-appraisal about when a problem is potentially solvable, given one’s current state of knowledge and experience. If, in the judgment of the solver, success is improbable even with extreme perseverance, it may be a frustrating waste of time and effort for the solver to persist.

Persevering by definition involves a choice, such as whether to continue problem solving or to give up. As with most choices, there may be opportunity costs associated with choosing one path over another. Deciding whether to persevere requires consideration of whether it is worth using a solver’s finite time and efforts to continue working on a problem, or alternatively (if eventual success is viewed as highly improbable) whether it might be more productive to give up. This is a difficult choice for a solver to make, requiring accurate metacognitive self-assessments of one’s knowledge and the likelihood that one may be able to achieve success with effort.

Note that my point is not that we should be more supportive of allowing students to give up. The idea behind a focus on perseverance—a belief that with learning come challenges, and with effort these challenges may be overcome—is very powerful and important. But at the same time, we also want to foster within students the ability to make (admittedly subjective and nuanced) judgments about whether they have adequate knowledge to make progress on a problem. If a student does not know enough, or is not able to monitor her progress on a problem, it likely makes more sense for the student to give up and perhaps consider returning to the problem when she knows more. Perseverance on tasks that are well beyond one’s capabilities could be viewed as a Sisyphean exercise.

Second, there is a fine line between perseverance and stubbornness, where the latter (in a problem-solving context) involves getting mentally stuck on a certain problem-solving path such that one has great difficulty moving on to different or new directions. **Perseverance that is actually stubbornness may be counter-productive to future problem solving success, even when it appears to be useful in the short term.** A hallmark of perseverance is a willingness to
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Persist in the face of initial difficulties, with the hope that with continued effort, success may eventually come. The current emphasis on perseverance may suggest to students that they should stick with the strategy or range of strategies that have found to work or potentially work in the past. Yet stubbornly sticking with a tried-and-true strategy—even one that has appeared to have been successful—may not ultimately be helpful. It may be better to abandon known or familiar strategies, even when there is some hope that these strategies could be successful in the short term. Giving up, moving on to other tasks, considering and developing strategies that are conceptually very distant from what one has used before, might actually be the most promising approach to ultimately completing a difficult challenge.

Taken in this light, perseverance (particularly as it veers toward stubbornness) can appear to be the opposite of what we aspire for students. Successful problem solvers are able to innovate, to adapt, and to be flexible. Successful solvers have a broad repertoire of strategies and a willingness to change, tweak, and optimize what they know, in order to try to make progress in completing a problem. Solvers often need to think outside the box in order to expand the current state of their knowledge, perhaps by asking questions such as, “Is there a better way to do this?” and “Does this strategy always work?”. Perseverance may suggest not giving up and sticking with what has worked in the past. However, it is much more difficult, and requires substantially more initiative, to decide to abandon one’s current efforts, including giving up or completely switching solution paths. Both Benny and Kathy in the anecdotes above exemplify this fine line: Both of these students appear to be stubbornly persisting, and in neither case is it productive.

It is also worth noting that the policy documents discussed above, including CCSS and NCTM’s Principles to Actions, send somewhat mixed messages related to the distinction between perseverance and stubbornness. CCSS states that students should “change course if necessary” without providing any specifics on how a solver should know when change is “necessary.” Related, Principles to Actions indicates that students should be willing to struggle. But if a student is using a strategy that appears to be effective but is not causing struggle, should she try a new approach that is more challenging albeit possibly less effective? In other words, should a student persist in using a strategy that works but is easy to use, or should a student risk changing course in order to seek struggle? And to which of these alternatives is the word perseverance most applicable? This is murky terrain, but the larger point is that when perseverance is actually stubbornness, it may no longer be a desirable goal for students.

Finally, another important nuance in considering when it is useful to persevere concerns solvers’ assessment of the utility of persisting. Especially considering the motivational aspects of perseverance, it seems clear that solvers should take into account the perceived value of a task...
and of completing the task, as it may not necessarily be worthwhile to persevere with every task. In order to be motivated to engage in a task, a solver must believe it to be worthwhile to engage in and/or to complete the task. Persisting in a task that is viewed as lacking value may seem pointless.

The importance of value is core to contemporary theories of motivation, particularly expectancy-value theory (e.g., Wigfield & Eccles, 2000). Expectancy-value theory suggests that we can understand motivation to engage in academic tasks by considering whether students expect to be successful in the task as well as the extent to which they find value in engaging in the task. Furthermore, this theory posits four subcategories of value, including interest (how enjoyable it is to engage in the task), attainment (whether the solver finds engaging with the task to be consistent with his or her self-image), utility (whether the solver finds engaging in the task to be useful for his or her future goals), and cost (the perceived cost of participating in the task). Each of these sources of value seem important, although some evidence suggests that attainment value and interest play particularly important roles in perseverance (e.g., Hulleman et al., 2008). When a student is not interested in a task and/or if she feels that engaging with the task is not well aligned with her self-image, motivation for persisting with the task may be lacking.

Both CCSS and NCTM’s Principles to Actions consider certain types of tasks when advocating perseverance—tasks that are ill-structured, rich, and have the potential to be cognitively demanding. There is perhaps an assumption that these types of rich tasks will be inherently interesting to students—more so than more traditional tasks such as the equation-solving exercise that Kathy solved in the anecdote above. Certainly interest does play an important role in determining whether students value engaging with a task. But other factors, relating both to the individual student and to the culture and communities to which that individual belongs, can also exert a strong influence on value, particularly attainment, utility, and cost value. Especially in light of these other factors, choosing ill-structured and rich tasks is neither necessary nor sufficient for ensuring that students consider persevering to be worthy of their time and efforts.

**Conclusion**

To summarize, perseverance has been identified as an important outcome for all students in various mathematics education policy documents, including the CCSS and NCTM’s recently issued Principles to Actions. Although perseverance is not explicitly defined, a close reading of these documents points to both metacognitive and motivational conceptions of perseverance. Although I am generally supportive of this mathematical problem-solving goal for students, in
this paper I have also introduced nuance that seems important to consider to perhaps temper our advocacy for perseverance. From a metacognitive perspective, the core issue underlying a decision about whether to persevere is whether one knows enough, including if the determination of whether one knows enough is even possible, such that one’s efforts have a reasonable or even slim chance for eventual success. If sufficient and relevant prior knowledge to enable productive engagement with the task does not exist, or if one is unable to make an informed judgment as to whether success on the task is even the realm of possibilities, it is not clear whether perseverance is a good use of one’s time and effort. From a motivational perspective, the choice about perseverance seems related to whether the task itself, engagement with the task, and completion of the task are all of sufficient value such that it is worth an investment of time and effort. If the task is viewed as uninteresting, and/or if one feels that the energy that might be devoted to the task is not useful personally or toward some larger enterprise, continued engagement with and investment in the task seems pointless.

One critique of my arguments in this paper would be a claim that perseverance implicitly carries within its definition the goal of knowing when to persevere and when to give up. According to this perspective, to persevere means to not give up as well as to possess the ability to make decisions about when to not give up; in other words, to persevere is to know when to (and when not to) persevere. Such an expanded (and, in my opinion, somewhat self-referential) version of perseverance is quite consistent with the points that I make this paper. However, it is worth noting again that I was not able to find clear definitions of perseverance and that I did not find hints of this more circular conception of perseverance within the policy documents that I examined.
References


