Beyond "You Can Do It!"

Developing Mathematical Perseverance in Elementary School

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Abstract

"Perseverance," an important psychological construct, matters for mathematics learning because solving challenging mathematics problems and reasoning about mathematical ideas often requires a kind of uncomfortable persistence. However, school experience often signals speed to be a marker of mathematical skill, and learners are rarely guided explicitly to see that perseverance is needed or how to stick productively with a tough problem. In this paper, we argue that perseverance is a domain-specific bundle of capacities and that it is not a trait, but can be deliberately taught and learned. We argue further that perseverance can be developed as a collective as well as an individual practice, and that collective work can help develop individual perseverance. We use a case of the teaching and learning of a challenging mathematics problem in an upper elementary mathematics class to illustrate and unpack these elements of the paper's argument. Our analysis focuses on three aspects of the instruction: the nature of the mathematical task on which the class was working, the sequencing of students' work on the problem, and how perseverance—for this problem and beyond—was supported. The paper concludes with several questions that our analysis suggests as important for next steps in trying to understand mathematical perseverance and what it takes to support and develop it.

"Sticking With It" in Mathematics

A common finding is that U.S. students tend to give up if they cannot figure out how to solve a complex problem within a few minutes. Stevenson and Stigler (1989) report, from a study comparing U.S. students with their counterparts in other countries, that, on average, U.S. students tend more than others to believe that mathematics depends more on talent than effort. Could this persistent cultural assumption be challenged? Could students be helped to develop greater willingness to persist when faced with a mathematical challenge? What might such explicit intervention entail?

Consider a group of eight-year-olds who are attempting to write equations for 10. One student writes 1 + 9, 2 + 8, and 6 + 4, and pushes her paper to the side and announces, "I'm done." Another sits, head on his hands, and says, "I don't get it." Two others, leaning close to a shared paper, have created a table and have listed all the two-number expressions (e.g., 9 + 1, 1 + 9 , 8 + 2, 2 + 8, etc.); once they are done, they ask the teacher if their solution is right. The teacher calls the class together and begins a discussion by showing them another solution that no one has written yet: 0 + 5 + 9 – 2 – 2 = 10. A murmur runs through the group, "Oh!!" "Can someone think of another possible equation?" asks the teacher. One girl ventures, "20 – 5 – 5?" "Does that equal 10?" asks the teacher. Several children nod vigorously. "Try to find some more equations now," says the teacher. "Do you have some things you can try?"

Although the students have not discovered this yet, this is a problem with infinitely many solutions. Over the next half hour, the teacher asks questions and pushes them gently to keep finding equations. She pulls the class together and tells them to put their papers in their folders, and that they will work to find more solutions tomorrow. The students seem excited to keep working and one asks when they will have time to work on this some more. The fact that the number of equations is endless might seem likely to provoke frustration on the part of the students. However, as they work further, their gradual realization that "the answers go on forever" both excites them and fills them with a sense of accomplishment.

In this brief episode, we see a simple example of supporting students to persevere with what, for them, is a complex mathematics problem. Why would these students persevere with this challenging task? One reason is that they find it interesting; a second is that they think they are capable of working on it. The judgments supporting each of these motivations are both intellectual and social-emotional. In particular, we argue that perseverance is neither a trait nor generic. The disposition and skill to stick with a difficult task depends both on context and on the specific arena of the challenge. One might be willing to persevere in training for a marathon but give up quickly when working with an impatient partner on a puzzle.

Few students develop the confidence and skills needed to persevere with challenging mathematics problems in school. Studies have shown that it is common for students to give up within a few minutes if a problem is confusing or a solution does not seem imminent. These same students often persist with other challenges outside of school—perfecting a jump shot, constructing puzzles, or practicing a difficult musical piece. In school, they develop the sense that mathematics is more a matter of talent and speed than persistence and effort. And when they are exhorted to try harder, rarely are they helped to learn what to do to stick with a difficult mathematics problem. Further, when they get messages that they are not good at what school calls “math,” they are even less likely to have the will to try to work on puzzling or difficult mathematics problems. What might instruction look like that would aim to help students, especially those who have been discouraged in school and who feel themselves not to be “good at” math, to have both the skills and the confidence to persevere in mathematics?

What Is Written About Perseverance in Mathematics?

One place to look could be to the psychological literature on perseverance. Researchers interested in what it takes for people to stick with challenging tasks, or to defer gratification long enough to stick with something, study this under natural and experimental conditions. Dweck (e.g., 1999, 2006) identifies the key importance of what she calls a “growth” mindset that shapes learners’ willingness to try because they believe that a difficult challenge can be mastered. She contrasts this with a common “fixed mindset” associated with the idea that capability is innate and not mutable. Persistence[1] has been shown to relate to the presence of “growth mindset” (Duckworth et al., 2007). Other researchers investigate the conditions under which children will stick with a puzzle or a task, or whether they can wait for a promised reward (e.g., Mischel, Ebbeson, & Raskoff-Zeiss, 1972). These studies do not, however, examine whether and how these behaviors shape learners’ persistence on a mathematics task in school.

In recent years, the importance of “sticking with it” has been foregrounded in specifications of K-12 mathematics learning. For example, the National Research Council report, *Adding It Up: How Children Learn Mathematics* (Kilpatrick, Swafford, & Findell,2001), highlights “productive disposition” as one of five key “strands” of what it takes to be mathematically proficient. They define this as the “habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (p. 116). They explain that “productive disposition” is

[1] We use both persistence and perseverance in this chapter, although our sense is that perseverance is the more appropriate term for what we are writing about. Persistence can connote a stubborn resistance to or a lack of change, whereas perseverance seems to imply a steadfastness to continue even through barriers. Because we are interested in the skills involved in sticking productively with challenging mathematical problems, we are inclined toward the word “perseverance.”

interdependent with four other “strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning.” Thus, seeing effort in mathematics as worthwhile and viewing oneself as capable of and effective at learning and doing mathematics are related to one’s understanding and skills. But capabilities are also developed as a function of having a productive disposition toward mathematics.

Having a productive disposition as represented in *Adding It Up* comprises attitude and mindset more than a set of specific skills. Cuoco, Goldenberg, and Mark (1996) unpack the idea of “mindset,” which they call “mathematical habits of mind,” into a set of core skills, techniques, and ways of thinking that are fundamental to disciplined reasoning—in general, in mathematics, and in specific mathematical fields. These authors argue for a fundamental reorientation of the school mathematics curriculum. Specifically, instead of an exclusive focus on facts and procedural skills, the school curriculum should “elevate the methods by which mathematics is created and the techniques used by researchers to the same status as that of the results of that research” (p. 376). These methods include intellectual practices (e.g., experimenting, sniffing patterns, visualizing, using multiple perspectives) situated in a set of basic dispositions toward the doing of mathematics. Cuoco and his colleagues thus focus less on beliefs of the learner and more on key skills and approaches that can be taught and learned.

Consistent with Cuoco, Goldenberg, and Mark’s (1996) position about the appropriate orientation for the K-12 mathematics curriculum, the Common Core State Standards Initiative (CCSSI) makes mathematical practices of equal status to specific mathematical topics (National Governors’ Association [NGA] & Council of Chief State School Officers [CCSSO], 2010). Like both Cuoco et al. (1996) and Kilpatrick et al. (2001), however, the CCSSI emphasizes the fundamental interconnectedness of specific mathematical “content” with ways of approaching that content. The first mathematical practice, to “make sense of problems and persevere in solving them,” is described in terms of what a proficient mathematics learner does to persist productively in mathematics. Included are strategies, such as trying a simpler version of a complex problem or considering similar problems, as well as metacognitive skills, such as monitoring and appraising one’s progress and changing direction when needed. Also listed in the dense text are resources such as using graphical calculators as needed or making relevant and useful diagrams. A major emphasis is placed on keeping track of whether one’s work is making sense given the problem at hand. In the Common Core, then, perseverance is also represented as something to do, not just an attitude.

(How) Might Mathematical Perseverance Be Developed?

In this paper, we analyze a case of instruction that aims to cultivate productive mathematical persistence in elementary students. Of interest in our research are learners who, on average, have not had opportunities or encouragement to succeed in school mathematics, especially with challenging and complex work, and who, by fifth grade, are already less and less willing to try problems that seem hard or confusing. Our analysis focuses on the contributions of the mathematical task, the teacher's role and practices, and the whole class discussions. The task in our study has an unusual feature that is particularly salient with regard to cultivating perseverance: it is mathematically impossible.[2] This predicament presents the children with a crisis because they assume that all problems that teachers assign have solutions. Their past experience has led them to think that if they do not find a solution, it is because they are not capable of solving it. But in this case they are able, with support, to compose, from the results of their work, a proof that the problem has no solution. The solution, which takes the class about a week to produce, provides a window into what is involved in students learning to persevere with a challenging mathematics problem.

We begin by explaining the context for the episode we use in this paper, and continue by unpacking the mathematics task used, both in terms of its mathematical features and the complexities that the problem presents for the students in this case. We then turn to describing the trajectory of the students' work on the problem. Following this, we present our analysis of the key features of the instruction designed to support these students' perseverance with this novel and challenging problem. We conclude by summarizing what we take from this case and suggesting questions for further investigation with respect to the cultivation of productive mathematical persistence.

Instructional and Research Context

Our investigation of mathematical perseverance took place in the context of a mathematics program, the Elementary Mathematics Laboratory (EML), for rising fifth graders that takes place at the University of Michigan every summer. The program enrolls approximately 30 students each summer, recruiting the children with the assistance of the partner school district, a working class, racially and ethnically diverse, district whose students live in two adjacent communities. A majority of the students are economically disadvantaged, and there is a growing number of homeless children.[3] Mathematics achievement is a broad concern in the district, as indicated by students' scores on the Michigan Educational Assessment Program. These data show that approximately 70% of all fourth-graders are "not proficient" (the lowest level on the state assessment program).[4] The target for participation in the

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[2] Mathematical problems without solution have a colorful and distinguished history (Suzuki, 2009) in mathematics (for example Fermat's Last Theorem), but we have found only one reference to their deliberate use with elementary students, in Burchartz and Stein (1998).

[3] Source: *2014 Report Card for Washtenaw County Children and Youth*.

[4] Source: Michigan School Data, *Annual Education Report 2012-13,* for the communities that comprise the EML partner school district.

EML is students who have not experienced success with school mathematics, including those who are having significant difficulty. Students are randomly selected for participation in the program, but the gender composition is deliberately balanced. The classes vary slightly from year to year, but typically about 75-80% identify as black or mixed race, and about 10-12% identify as Latino/a. Most of the students who apply for the EML have gaps in their skills and knowledge and typically have not enjoyed or felt confident with school mathematics. Although a strong effort is made to recruit students who have not been doing well, the EML is not designed or advertised as a remedial program, but seeks instead, at the same time as filling students' past gaps, to advance them into more challenging and complex mathematical work.

In the EML, the children work on mathematics for 2½ hours each day for five days across two weeks. The mathematical content of the instruction includes work on fractions (definitions, representations, placement on the number line), as well as on reading, interpreting, and solving equations. The students also encounter and are supported to solve complex and unfamiliar mathematics problems. Mathematical practices and techniques include explaining, representing, proving, presenting in public, and listening to others' mathematical ideas attentively, respectfully, and critically.

Our research group has been designing and studying the teaching and learning of mathematics in the EML for over ten years.[5] Some of our research involves deliberate design and study, whereas some involves us in analyses of what happens. Our investigations include analyses of a wide range of components of the instructional context. Examples of studies include the representation and treatment of the number line as a mathematical object, the teaching and learning of reasoning and proving (Ball & Bass, 2000, 2003), the design and use of the daily student homework (Ball, 2008), the nature of instructional resources that support the inflight work of teaching (Shaughnessy, 2012), the teaching and learning of specific mathematics learning practices (Goldin, O'Neill, & Naik, 2014), instruction that simultaneously fills in gaps and also stretches and challenges students (Mann & Owens, 2012), and the relational work of teaching across ethnic, racial, and gender differences (Noel, 2014). Clearly, given the students' past school histories with mathematics, their persistence is a key issue. They are not, on the whole, confident and when they encounter unfamiliar problems, the children tend to say that they cannot work on them because they do not understand or have not been taught the material.

[5] The EML program is also designed for collective close study of instruction by a variety of professionals, including teachers, education researchers, teacher educators, mathematicians, and education leaders who are in residence for different lengths of time during the program. To this end, the instruction is conducted in a "fishbowl" format, with an audience of professional observers seated in a "gallery" behind the classroom, or watching in a remote viewing room where observers can see and hear what is going on in the classroom with video feeds from two cameras. Moreover, high quality comprehensive digital records of the instruction, including student work, lesson plans, and other artifacts, are gathered and made available for later study and analysis.

Therefore, a goal of the program is to increase students' skills with, and actual experience of, perseverance. Working on this has required the team (led by Ball along with Meghan Shaughnessy and, earlier, Laurie Sleep) to develop ways to create opportunities for students to develop skills of persistence.

Opportunities to develop perseverance in mathematics. Across the two weeks, students encounter multiple tasks and intellectual demands that are different from their past school experience. For example, they are expected to speak in class, both in considerate response to their classmates' ideas and in order to present their own. They are asked to go to the board and explain their ideas and solutions. They learn to use new representations, such as area models for fractions and the Minicomputer (Papy, 1970). Solutions and ideas must be accompanied by explanations and justifications, a requirement that is mostly new for them. As they engage in this range of work, the teacher explicitly names particular things to try and supports the students in attempting these novel tasks and expectations. For example, in helping students learn to write mathematical explanations in their notebooks, the teacher interactively models possible alternative ways to express and record justifications. To make visible what is involved in using the area model to represent fractions, large replicas of two diagrams are drawn on posters and students try to express the key features of the different representations. As different students attempt explanations, the teacher works with the children to extract and name aspects of the diagrams to explain the mathematical ideas (e.g., identify the whole, check whether the whole is divided into equal parts, determine what one of the d equal parts would be named [1/d]). Together the students and the teacher construct protocols for certain fraction explanations and publically record the new learning in ways that support them to try explaining the succeeding examples. They are not just exhorted to "try harder" or cheered on ("you can do this"). Instead, through public talk, unpacking with highlighting, naming, recording, and collective practice, students are helped to persevere with difficult challenges.

In addition, two specific experiences create explicit opportunities to develop mathematical perseverance. One involves the nightly homework (Ball, 2008; Shaughnessy, Ball, & McNamara, in preparation), on which the task types ("practice," "extend," and "you be the teacher") include one called a "looking ahead" problem. The students are told that the "looking ahead" problems involve ideas that they have not yet worked on in class. They are asked to try to make sense of what is being asked and make an effort to at least start trying to solve it. These problems involve material that will be worked on within a few days, and so the students get a little preview and also get to see what they can make of what is being asked. The goal is not to solve or even complete these tasks, but simply to learn to not shut down ("We haven't done this") and to stretch a little.

The second opportunity involves a complex multi-step problem that actually has no solution; the goal is for students to succeed in proving that no solution exists, a proof of impossibility. It is on this task and the surrounding instruction that this paper focuses.

The Train Problem as a Context for Developing Mathematical Perseverance

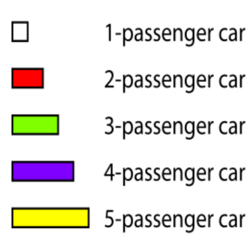
One reason for using an impossible problem is that the students assume that all problems in school are solvable, and if they are unable to solve a problem, it means that they lack capability or have failed. This experience is intended to build their confidence about the nature of the solution set for a given problem, and is connected to other experiences they have with problems that have one, multiple, or infinitely many solutions. A second reason is that the problem is challenging and cannot be solved, at least not by fifth graders (and probably many adults) in one sitting. Working on this task thus offers occasion to learn specific strategies for persevering with a large and confusing problem.

The mathematics problem. The basic underlying mathematical task is to find an order in which to list the five numbers 1, 2, 3, 4, and 5, without repetition, in such a way that when subsets of adjacent numbers in the specific list are added together, every number, from the smallest to the largest is possible. For example, one such order is:

2 3 1 4 5

In this arrangement, the largest number, 15, is possible by adding all the numbers together. Six is possible by adding 2, 3, and 1 together, which are adjacent this list. Ten can be made in two ways, using 2, 3, 1, and 4 or using 1, 4, and 5. But 12 is not possible because the numbers needed to add up to 12 using subsets of 1, 2, 3, 4, and 5 (3, 4, and 5 or 1, 2, 4, and 5) are not adjacent in this arrangement. Similarly, 14 is not possible.

The actual task given to the students is contextualized in a fanciful scenario called the “train problem” and makes the class into an imaginary “EML train company” that constructs “trains” to order. This formulation of the problem uses Cuisenaire rods to represent cars on a “train.” A “train” is a structure built by laying the rods end to end. The number of passengers that the “train” can hold is determined by summing the number that each car in the train can hold:



A train composed of exactly one red, one green, and one yellow “car” would thus hold 10 passengers (i.e., 2 + 3 + 5). The problem is formulated with a story about a customer, Mr. X, who seeks a very special train from the train company: Mr. X wants to order a five-car train that uses one of each of the different-sized cars. He also wants to be able to break apart the five-car train to form smaller subtrains that hold fewer passengers. To make things tricky, he wants to be able to form these subtrains only using cars that are next to each other in the larger train. And what he wants is one train that makes it possible to form subtrains for every number of passengers between 1 (just the white car) to 15 (all five cars). Although every possible train would make it possible to build a subtrain for 1 passenger (use only the white car) and one for 15 passengers (use all the cars), it is not clear that you can find a subtrain for every number of passengers in between.

This task is structurally equivalent to the basic numerical version of the task described above, but the context adds several features designed to support engagement in this complex problem. One feature is the concrete materials for building the trains. A second is the storyline of a demanding customer who orders this very unusual train, and a “train company” that is trying to satisfy the customer’s request. A third is that the customer, Mr. X, grows increasingly impatient, and his relentless question, “Is my train ready yet?” provokes an effort to find a solution.

Mathematically, although the “math” seems to entail just simple addition combinations, this problem is far from routine. Most adults find it challenging. One source of the challenge is the scope of the problem. With five cars (or numbers), there are 120 different arrangements.[6] This is too many trains to build and check one by one. Furthermore, each individual train requires up to 15 checks. For example, consider this train:

1	**3**	**5**	**4**	**2**

This order does make it possible to construct smaller trains that hold 1, 2, 3, 4, 5, or 6 passengers but not 7. It does allow trains that hold 8 or 9 passengers, but not 10. Moreover, as soon as one passenger number (of a subtrain) is discovered to be impossible, the particular train can be rejected because it does not satisfy Mr. X’s request. But before throwing out this train, the claim—that subtrains that hold either 7 or 10 passengers are not possible—must be proved. This takes some careful reasoning, either by exhausting all combinations or through some other reasoning. For example, one could prove that 10 is impossible because the whole train holds 15 and there is no way to remove a car or cars that hold a total of 5 passengers (i.e., 15 – 5 = 10) that also leaves the remaining cars connected.

The sequencing of the students’ work on the problem. The students’ work on the train problem stretches over several days, spending about 45 minutes to an hour each day. Although they do

[6] Consider how to see that there are 120 different possible trains. For the first car, each time there are five choices, for the second, four choices, for the third, three choices, for the fourth two remaining choices, and then for the fifth car, just one is left. Thus, the number of different trains one can build is 5 x 4 x 3 x 2 x 1 = 120, which is also called 5 factorial and written as 5!.

work on parts of it individually and with partners, the problem is presented as a collective task, and the challenge of solving it is the responsibility of the group (i.e., of the EML Train Company). The trajectory proceeds through six stages: (1) becoming familiar with the context; (2) collectively making sense of and interpreting the problem, and identifying conditions (or mathematical constraints); (3) building and checking trains and recording results; (4) confronting the scope and feeling discouraged and overwhelmed; (5) cutting the problem down to manageable size; and (6) proving and becoming confident that no solution exists. The teaching across these stages makes explicit specific practices of tackling and persisting with a difficult mathematics problem. At each stage, these practices are highlighted and labeled, modeled, scaffolded, and rehearsed and refined. In what follows, each of these teaching moves is explained and illustrated.

Becoming familiar with the context. The students are first introduced to the territory with the task below. It is designed to help them to uncover some of the key features of the context and to practice reasoning about trains:

> The EML Train Company makes five different-sized cars: a 1-person car, a 2-person car, a 3-person car, a 4-person car, and a 5-person car. These cars can be connected to form trains that hold different numbers of people. Try to build some trains. You can use <u>only</u> these five types of cars to build trains, and you can use <u>at most one</u> of each type of car in each train.
>
> What are the different numbers of people that the EML Train Company can build trains to hold?

As the students explore, build, and discuss the trains, they are able to conclude that the greatest number of possible passengers is 15, and that a 15-passenger train requires all of the cars. They also figure out the least number of passengers possible is either 0 (no cars) or 1 (only the white).[7]

During this stage, the teacher emphasizes the importance of finding a system to record and organize solutions in a way that the solver can understand and use. One issue has to do with how to record trains. With letters denoting the colors (i.e., p + g + r)? With equations (i.e., 4 + 3 + 2)? By actually drawing and labeling individual trains? The teacher models on the document projector different ways that students have found, and engages the class in discussing the affordances and limits of different methods of recording. She does not privilege any single approach, but is deliberate about the importance of recording and makes different methods available and usable.

A second core practice that underlies persistence is gaining sufficient familiarity with the materials and the context. The teacher comments several times about the importance of not rushing but of making sure one is comfortable with the terms and materials.

[7] It is interesting to listen to fifth graders debate whether a train with no cars is in fact a train.

Both these practices are made explicit by labeling them and by talking about the objects and the reasoning. Key terms (like "only" and "at most") and ideas are recorded on posters so that the language becomes collectively and subsequently usable.

After a period of working on how to build trains for different numbers of passengers, itself a challenge that requires considerable organizational and recording skill, the students are given the problem with the story of the customer:

> A customer named Mr. X wants to order a special five-car train that uses one of each of the different-sized cars. He wants to be able to break apart his five-car train to form smaller trains that could hold exactly 1 to 15 people. In addition he wants to be able to form these smaller trains using cars that are next to each other in the larger train.
>
> Can the EML Train Company fill the customer's order? Explain how you know.

Making sense of the problem and identifying conditions. The teacher leads a discussion of what the problem is asking. After a student reads the problem aloud, the teacher asks who would like to explain what Mr. X wants. Several students offer ideas. The teacher invites someone to come to the board and build one train (using magnetic Cuisenaire rods that stick to the white board so that everyone can see). Together they work on interpreting what the words in the problem mean, experimenting with what it means to "break apart" a train and to "form smaller trains" with cars that are next to each other that hold other numbers of passengers. For example, in working with one possible train,

students practice showing that smaller trains that hold 1, 2, 3, 4, 5, 6, and 7 are possible. They are stuck trying to find a smaller train that holds 8 passengers. The teacher asks different students to explain how they are sure that they cannot make a smaller train that holds 8 by breaking apart the whole train and, using cars that are next to each other.

As part of this stage of the work, the teacher and students extract and publically record the "conditions" of the problem. The idea of "conditions" is a tool that has been previously developed as part of learning to make sense of mathematical problems and setting up to check and justify possible solutions. In this case, the conditions that they identify, in their terms, are:

1. Only use w, r, g, p, y;
2. Must use all of these cars;
3. Must use each car exactly once; and
4. Must be able to form trains for every number of passengers from 1-15 without moving cars around. These smaller trains must be built from connected cars.

Building and checking trains and recording results. The students start trying, more or less randomly, to build a train that satisfies the conditions. Quite often in the early stages, students will call out, “I found it!” Sometimes a quick group meeting is called to hear the report and to check, collectively, whether the proposed train does in fact meet Mr. X’s requirements. The most frequent misstep is to violate the fourth condition—that is, to make different subtrains without keeping the order fixed. Practicing using the poster with the conditions builds up students’ capability with the problem.

One important practice to support perseverance with the problem is finding a usable and interpretable method of recording one’s work. Initially the students record their solutions in their notebooks (see Figure 1).

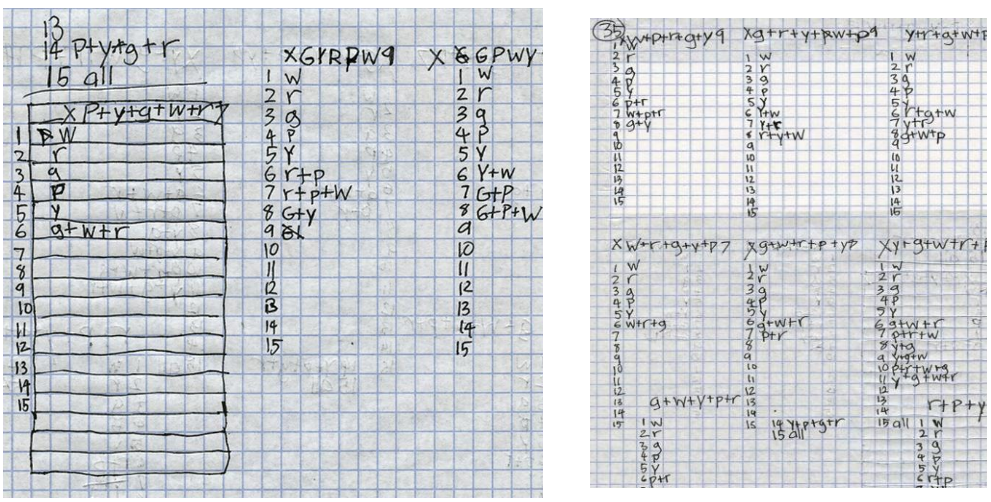


Figure 1. Two students’ recordings of their efforts to find a train to meet the customer’s order.

As they grow increasingly intent on finding a train that meets Mr. X’s conditions, their recording often becomes sloppier and less complete, and few students record in clear detail. The lack of organization and means of keeping track begins to present a barrier to the children’s progress, so at this point, the teacher introduces a blank chart for recording, e.g.:

TRAIN	SMALLER TRAINS' PASSENGERS	YES/NO
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	
	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15	

In this chart, students record the specific trains they build (using whatever system of recording trains that they prefer) and on the right, they circle the numbers that are possible for some connected subtrain of that specific train, and cross out those that are impossible.

As necessary as recording is to succeeding with a problem of this scope, a different important skill of persistence at this stage of the work is to eliminate unnecessary work. A challenging mathematics problem can easily become even more overwhelming if one spends time and effort on some aspect of the problem that does not improve one's progress with it. For example, the teacher asks at one point for comments about what "we know" about the problem. Students share ideas about where the class is, and one important finding is that it is not necessary to check for 1, 2, 3, 4, 5, or 15 because subtrains for these numbers are possible with any arrangement.[8]

Being able to periodically stand back in this way and appraise what one has accomplished is an important skill for persevering with challenging mathematics problems. For example, many students continue checking a train even after they have established that one passenger number is not possible for any subtrain of that train. Instead of pointing out to students the fact that they do not need to keep checking, the teacher deliberately makes this process of thoughtful monitoring transparent. To do this, at one of the periodic group reports during which students share trains they have tried, she leads the class in checking multiple subtrains even once one has failed. When she asks for comments from the class, a student objects to the continued checking of other subtrains. When the teacher asks for a reason, the student explains that, "we already know that this train doesn't work, so we don't need to keep checking it."

Confronting the scope and feeling discouraged and overwhelmed. For a while, students work feverishly to find a train that will satisfy Mr. X. Gradually, after a couple of days filled with many false alarms about finding a train "that works," eliminated by collective inspection, the students begin

[8] A subtrain to hold 1, 2, 3, 4, or 5 passengers can always be made from any train simply by using the appropriate single car to hold that number of people (i.e., the white, red, green, purple, or yellow). The whole train always holds 15.

to grow discouraged. They find themselves trying trains that have already been built and that did not work. The problem appears endless. The students have hit a barrier and their interest and determination, collectively, begins to flag. They do not seem to be making progress.

***Cutting the problem down to manageable size*.** What the students do not notice is that they have fallen into working from 6 up. With every train they build, they have sought subtrains for 6 passengers, then 7, etc., until they reach a passenger number for which no subtrain can be made. The teacher watches for an opportunity to bring this pattern into view. To support the students' progress, she could just suggest trying to work down from the big numbers—that is, to try to make 14 first, rather than to move upward starting with 6. But the goal is also for students to learn the general principle that when one is stuck on a math problem, it is useful to see whether there is another way to work on it. In particular, when a problem appears endless or too big, it is useful to step back and figure out whether there is a way to reduce the scope or space of their work. Sometimes this might mean trying a simpler version of the problem. At other times it might mean considering whether one is stuck because of the way one happens to be working. In this case, the fact that the students are always starting with 6 means that they are not coming to see an important aspect of the structure that leads to a major reduction in the problem's scope. To represent this kind of thinking, on the third morning, the teacher reports that Mr. X has called and asked, "Why are the students always starting from the small numbers?" This prompt both pushes them forward but also calls attention to the fact that shifting the approach can help them feel less stuck.

The students begin trying to check from the largest possible subtrains—that is, for 14 and 13 passenger subtrains out of the trains they build. After about half an hour, a student excitedly ventures a conjecture, "To make 14, the white should be on the end." The teacher stops the class, and asks them to listen to their classmate's conjecture. "What is he saying?" she asks. Through careful questioning and asking students to listen to and comment on one another's ideas, the teacher supports them to restate his idea that to make 14, the white "should" be on an end. Understanding the conjecture is not equivalent to believing it or appreciating fully its contribution to solving the problem. The teacher asks them to figure out if it is true: "Is it just that the white should be on the end or is there a way to make 14 without having the white on the end?" There is a glimpse ahead of a major clue to push through the huge and overwhelming problem.

The students begin trying to build trains that do not have a white car on the end and from which a subtrain holding 14 passengers can be made. As before, a few students call out, excitedly, "I found one!" But because the class is trying to solve this problem as a group, some students quickly ask to see the train. They are able to uncover that the students have shown that a train with a white on the end *does*—not *does not*—allow a subtrain that holds 14 passengers. Through examples such as this where students collaborate rather than compete, perseverance is represented as something that can be enacted by a group engaged in a common pursuit.

After some more investigation, the students become more sure that to build the special train that Mr. X wants, the red car (2 passengers) has to be on one end, to make a 13 passenger subtrain, and the white car (1 passenger) must be on the other. The teacher invites a class discussion to try to convince themselves as a group that this is the case. They are able to establish that a subtrain that holds 14 passengers can only be made by being able to use the red, green, purple, and yellow cars together, which means that the white has to be on an end. Using similar reasoning, they show that, to build a subtrain that holds 13 passengers, the red car has to be on the other end. Although proving this is a huge breakthrough in reducing the problem, the students do not immediately or automatically see this. "What do we know now? How does this help?" asks the teacher. The students call out, "Mr. X's train has to have the red car on one end and the white car on the other!" The teacher asks them how many trains there are that meet that condition. "What cars will be in the middle?" she asks, and the students respond, "Yellow, purple, and green!" The teacher reminds them that they already worked on a warm-up problem a few days earlier, to figure out how many different trains can be made from these three rods. They leaf through their notebooks and find the solution for this problem: 6 different trains out of those three colors. This experience makes plain that, in mathematics, solutions from prior work or related problems can be useful. Although the permutations work they had done was not originally connected to the train problem, their work to prove that any three things (Cuisenaire rods, numbers, letters, etc.) can be rearranged in six different ways, using each thing exactly once. This turns out to be directly relevant. The students are able to make progress in solving this problem at this point by making productive use of this other work.

Proving and becoming confident that no solution exists. What remains next is to build the six trains that have white on one end and red on the other. These then have to be checked to see whether any of them meets the conditions of the customer's order. The teacher distributes this work to the class by assigning one train each to six small groups of students into which the class is divided. Each group must figure out whether its train fulfills the conditions of Mr. X's order, or, if it does not, be able to show where it fails. The class works eagerly on the newly reduced problem. There is a palpable sense of collective work—it is the class that is seeking a solution to the customer's order, not a race among individuals.

A day later, the class assembles to learn the results of the small groups' work. Soon it is apparent that no train has been found that satisfies the conditions. What does this mean? Some students feel confident, and ready to report to Mr. X that there is no train that can do what he wants. But others are disappointed and feel that the class has failed to find the train. This question represents the final hurdle in completing the problem. Is it possible to determine that the class has actually finished the work and that the answer is that there is no train? What does it take to persevere to the point of certainty? This is especially difficult in this case, where the "answer" is that there is no solution. To some students, it still seems that they have not finished the problem because they have not identified a train that meets the conditions. Others feel discouraged, and think the class has failed. Although it is not mathematically unusual, the special case of a proof of impossibility presents a novel challenge to these ten-year-olds.

Although the students know that Mr. X is not real, they are nonetheless caught up in the story of being a train company and trying to meet the customer's request. Figuring out whether they could convince him that there is no train represents one more skill in persevering with a difficult problem: the generic question of asking oneself, how would I prove this to a skeptic? Could I convince someone who did not believe me?

The teacher has the class prepare a presentation to Mr. X about the work they have done and the conclusion they have reached. They prepare to explain the steps of their work over the several days and rehearse the presentation. The teacher asks skeptical questions, playing the imaginary customer. "I'll just ask another class to make my train," she says. Stepping out of her role, she asks, "Is there anything we could do to prove to him that this would not work?"

The device of pretending to explain their work to Mr. X affords an opportunity to review the steps of their work as they stuck with a tough problem, as well as to certify and explain the conclusion. The "us and him" nature of the context supports the work as a collective challenge, in which resources and progress are shared and distributed. Persevering is represented as profiting from working with others who can provide ideas, notice things, and help get work done.

Building Mathematical Perseverance

Developing the capacity to persevere with challenging mathematics certainly requires opportunities to contend with puzzling problems where the solution paths are not obvious. But simply getting stuck in math class does not lead to productive perseverance. What is the nature of tasks that can engage students in productive struggle, rather than frustration?

Tasks that demand appropriate levels of perseverance. For the fifth grade students in the instructional case above, the train problem presented such a challenge. The mathematical ideas and skills involved were familiar (i.e., adding combinations of single digit numbers, writing single digit numbers in a particular order), but the question posed was complex. Its scope was large, too big to make it feasible to write down all the possible trains. So although students could begin, they soon hit a barrier that they did not know how to surmount.

Solving problems such as this one depends on persistence. But students do of course regularly confront problems in school that are far from simple or obvious to them, and these do not necessarily build perseverance. In fact, students confronted with obstacles often give up. Of course some develop robust skills and ways of persisting productively. But many do not.

One reason that encounters with challenging problems do not automatically build perseverance is because the problems are not designed to create opportunities to learn productive ways of persevering; they are simply difficult and frustrating. If students do not understand the terms or ideas involved and have no available pathways or resources to break through, their experience is more likely to be discouraging and will not engender determination. A second reason that challenging tasks do not automatically build persistence is that students are not explicitly helped to develop specific skills of perseverance. Being exhorted to "stick with it" or to "try" does not help students learn what they can do to make headway and get over a barrier or tackle a puzzle. Students already persist in other domains (e.g., assembling a Star Wars minifigure, building a puzzle, or perfecting a jump shot), but these experiences do not equip them with relevant skills to stick with hard math problems. So what is involved in helping students develop practices of mathematical perseverance?

Supporting the practices and skills to persevere with mathematics. When students struggle with challenging mathematics problems to the point of giving up, teachers have a number of options. One is to reduce the complexity of the work by giving students a simpler problem or otherwise adjusting the task. A second is to provide specific hands-on help, closely guiding students to proceed with the work. This help can be in the form of direct procedural guidance and doing the work with and for the student, e.g.:

> Okay, let's look at this train. *Teacher makes a train*.
> Let's see if we can make a smaller one that holds 6 passengers. *Teacher shows how to make 6.*
> Seven? *Teacher makes a subtrain that holds 7 and directs student to record that 7 is possible, and continues by testing each possible number of passengers until she identifies one that can not be done, shows the student and then crosses off this train on the record sheet to show that it fails.*

These common sorts of help often reduce the complexity of the mathematical task and change what students have to do by "degrading" the entailed cognitive demand (Stein, Grover, & Henningsen, 1996). A modified version of this form of teacher guidance is one in which the teacher is leading the work but asks questions throughout (*"Can we make a smaller train that holds 6 passengers?"*). In common, however, is that the teacher's support is directed at helping a student stick with this specific problem. With close support of this kind, a student might—and often does—persist with the problem at hand. Working with their teacher, students' work can be directed, guided, or scaffolded. If the problem is within what is sometimes called the "zone of proximal development" (Vygotsky, 1978), then students' participation with the teacher can deliberately extend their capability to accomplish the task (Brown, Collins, & Duguid, 1989). When it is beyond their reach, the work may be more like being led or dragged through the task, but without much actual learning. Perhaps students develop the confidence that they can solve hard problems from such experiences in the same way that a young child develops confidence to swim from wearing inflatable "floaties" on his arms in the swimming pool, or to ride a bike from using training wheels. But there are no assurances that students are learning the more general skills—for swimming, bike-riding, or solving hard math problems—from being supported to get the task done.

A distinctly different kind of instructional support is to make visible the techniques, strategies, and tools that are in general useful to accomplishing a complex practice. In the case of swimming, it would be to teach floating or kicking. In the case of bike riding, it might be to teach how to get on the bike and not fall down right away. In the case of challenging mathematics questions, what are the more general skills of sticking with hard work that help students build the capability to stick with tough problems, not just solve this particular one? And what are the instructional strategies or moves to support learners' development of these practices and the ability to use them independently?

In the students' work on the train problem, the teacher's role was crucial to keep them engaged with and moving forward with the problem. But some moves by the teacher helped support the students to stick with this specific problem while others made explicit a more generalized skill or habit for tackling challenging problems. For example, working explicitly on reading and interpreting the problem and what it was asking, practicing with a sample solution, and then listing the problem's conditions all afforded an opportunity for students to see what to do in general to figure what a problem means and what a solution would require. However, in contrast, concocting a "suggestion" from Mr. X helped the students discover that the white car has to be on the end helped the students break through a barrier. They needed to notice that they were always starting with the lowest numbers of passengers and thus not coming to discover that a subtrain for 14 could only be made with the white car on one end, which is a crucial insight for reducing the scope of the huge problem. But although this experience might show them the value of reducing a large problem, this instructional move would not help them learn how to do this with other large complex problems.

In this case, the teacher was the one who reduced the scope of the problem for the students by setting up prompts for them to come to see that the white and red rods would have to be on the ends. The mathematical insight that underlies this move is produced through a more general habit of reflecting on one's work on a problem and noticing how what one is trying or doing may be impeding progress. Here, the accidental habit of beginning the checking process for a given train by testing for the subtrains that hold few passengers instead of starting with the largest subtrains (14 and 13) got the students stuck. The teacher is the one who took this step, but did not model her thinking explicitly enough for this to be visible to students. As a result, this step opened up a pathway for them with this specific problem, but probably not beyond it.

Stage of the Work	Learning Goals for Students	Instructional Moves and Steps	Supporting Students' Progress on This Problem	Supporting Students' Learning More General Practices of Perseverance
1. Become familiar with the context.	Practicing reasoning about trains; practice with explaining.	Have students work on a simpler problem that establishes the conventions for building trains.	✔	
2. Collectively making sense of and interpreting the problem, and identifying conditions.	Learning strategies for making sense of and interpreting the problem; identifying conditions of a problem.	Engage students in reading and discussing the problem.	✔	✔
		Lead students in practicing one possible solution with one train.	✔	✔
		Guide students to extract the conditions of the problem.		
3. Building and checking trains and recording results.	Using conditions of a problem to check solutions; making records of one's work on a long problem.	Have students work on building trains and making records on their own.	✔	✔
		Encourage students to check solutions using the core constraints of the problem.	✔	✔
		Have students report out on specific trains, creating an incentive to organize records of trials.	✔	✔
4. Confronting the scope and feeling discouraged and overwhelmed.	Using feeling of "stuckness" to seek ways to simplify problem.	Allow students to begin to wallow in the enormity of the problem and their lack of success, stimulating desire to find a simpler way through it.	✔	
5. Cutting the problem down to manageable size.	Investigating a conjecture and seeking to prove or disprove it; using the results of prior work, seeking connections to other mathematical ideas.	Invent the customer's "suggestion" about the red and the white on the end.	✔	
		Refer students to prior work on permutations of three objects.	✔	✔
6. Proving and becoming confident that no solution exists.	Linking the imperative to convince someone as an imperative for mathematical proof.	Guide students to practice presenting to the customer that his order cannot be filled.	✔	✔

Table 1. Instructional moves and learning goals across stages of students' work on the train problem.

Table 1 summarizes the instructional moves and steps used in guiding the students' effort to solve the train problem. Some moves seem more likely to have promoted students' learning of general skills of mathematical perseverance than others. We turn next to considering the distinguishing features of instruction that seemed to promote the development of mathematical perseverance in students rather than only supporting them to solve the train problem.

Teaching for Mathematical Perseverance

Persevering with challenging tasks is clearly fundamental to skilled use of mathematics to solve problems. Successful problem solvers use a range of strategies that are often not made explicit to others (Schoenfeld, 1985), and the development of the capacity to persist productively is not usually a deliberate part of the school mathematics curriculum. However, recent learning goals for students make explicit its importance. For example, the first mathematical practice standard in the Common Core State Standards for Mathematics stipulates the goal of learning to “make sense of problems and persevere in solving them” (NGA & CCSSO, 2010) . In the description of this standard, many specific techniques useful to persevering are named, from explaining to oneself the meaning of a problem and ways to get started on solving it, analyzing constraints and givens, considering related problems, making tables or diagrams, and checking whether one’s work makes sense as one proceeds. A number of other techniques and habits are identified. Making the development of mathematical perseverance a fundamental practice raises the question of how students can be helped not just to persist with a particular problem, but to develop skills and techniques to persevere with challenging mathematics in general.

As students worked on the train problem, they were helped to engage in many of these skills and techniques. As a group, for example, they began by reading the problem aloud and discussing what it was asking. They tried an example train right away and used it to unpack the question of Mr. X’s order. Based on that, they extracted the conditions of the problem. As they proceeded, they frequently stopped to discuss together what they were finding, and regularly referred to the conditions of the problem and the central question. After a while, they were helped to be more systematic in keeping track of their experiments (Lampert, 2001). The students also used work on a related problem (permutations of three things) when they needed to build the trains that had red and white on the end.

We argue that for students to develop skills of mathematical perseverance requires both engagement in situations that require it and deliberate instruction. Such instruction can be usefully understood as a form of cognitive apprenticeship (Collins, Brown, & Newmann, 1989), in which the teacher makes visible and supports the development of the techniques, habits, and skills used in persisting with a challenging problem, and persevering through the barriers that arise in the course of doing mathematics. In “apprenticing” learners to the mathematically productive strategies that help to stick with a problem when one is stuck, it is important to situate the opportunities to learn in situations that call for perseverance and to support learners in seeing and doing, alone or with others, the work involved (Brown, Collins, & Duguid, 1989).

The train problem as a context for learning to persevere. As the fifth grade students worked to solve the train problem, the teacher used a variety of instructional strategies to help them persevere with the problem and to develop the capacity for mathematical perseverance more generally. First, she designed a problem context that required setting the group up to work collectively, instead of creating a problem to be done by the students individually. The context positioned the students as a train company with a customer who was placing an order to them as a group. Tackling this problem required public and shared engagement in the necessary strategies. Students were not left on their own to struggle, but instead contended with the challenging problem together. Efforts made by members of the class were publicly shared and discussed, and successful steps forward were everyone's accomplishment. This supported group discussion of ideas, results, and collective support for working on the problem.

Because this problem could not be solved in one fell swoop, perseverance was a distributed resource for the work of the group. From one day to the next, students described their progress in the first person plural, "We don't want the customer to come back yet," or "We're making progress." This "we" positioning was supported by the task of reporting to the customer, or preparing to explain their result to him. The collective positioning of the work not only enabled the need for perseverance to be a shared endeavor, but also created many occasions for externalization and explicitness through public discussion of moves, techniques, and strategies as the group sought to make progress.

Making practices of perseverance visible and learnable. The collective challenge of trying to fill Mr. X's order required frequent group discussions to check possible solutions and to consider strategies. As such, context created a situation that readily supported making the work visible. Although the playfulness of the story was obvious, and no fifth grader believed it was "real," the collective engagement and motivation to solve the problem nevertheless created a kind of authenticity.

As she supported the students in developing skills of perseverance, the teacher's instructional moves helped to make explicit the specific things that she and the class were doing as they pursued a solution to the problem. One significant move was *naming and labeling* particular tools for mathematical work (Ball, Lewis, & Hoover, 2008). Examples of this during the train problem included "conditions of the problem" and creating "systems" for "keeping track" of the trains tried. Having the problem's constraints labeled as "conditions" provided a verbal referent so that students could systematically and reliably test trains that they tried. The conditions were even posted on a chart in the room, and other examples of labeling and naming were also publicly visible on posters. The idea of "keeping track" and having a "system" for recording also helped make visible a way to maintain one's bearings in a large problem.

A second pattern in the teacher's efforts to make visible the practices specific to persevering with the train problem was the way she tended to *highlight and underscore* things that she or students did, calling attention to what was being done and talking about why it was helpful. Sometimes this talk was co-constructed with the students, but the teacher determined what needed to be highlighted and led the class in highlighting and explaining it. One example of this involved calling attention to how a student explained what the problem was asking and gave a non-example of a train, one that would not work. The teacher emphasized how useful it can be to try a solution that you think is not right and then see how that helps you understand the problem better. Another example of highlighting occurred when the students figured out that needing to have the red and white cars on the ends, the teacher highlighted the huge step that this was by reducing dramatically the number of trains that needed to be tried. Being able to reduce a problem is a generally useful thing to try to do in persevering. The teacher highlighted other techniques, such as particularly effective systems for recording trains that students developed (using letters, numbers, or other methods), clear explanations in which students traced their reasoning aloud for others to hear (she made the point that tracking your own work and thinking can really help you with a hard problem), and close inspection of proposed solutions (she made the point that care in checking solutions can be important to continuing to understand a problem better and better).

A third technique was a move that seemed to be a kind of *co-sponsoring* of mathematical work. As she saw individual students trying or doing things that could be of general interest or use, she would work quietly with an individual (or sometimes a pair or threesome), helping the student to extend and further articulate the idea or technique to a point where it could be shared with the class. One example of this happened when she noticed a pair of students crossing out 1 – 5 for each train they were checking. Through some questioning she helped them formulate their awareness that subtrains that carry 1, 2, 3, 4, or 5 passengers never need to be checked, and why. At this point, they were ready to propose this to the class for a broader discussion and eventual shared knowledge. The teacher co-sponsored other ideas that she saw students developing, helping them to express and explain but not ratify them, and prepare them for presentation to their classmates. By doing this, the teacher was able to bring a broader range of strategies and ideas to the class, and help all the students participate in developing ideas that were emerging as individuals worked. Once ideas were made part of the class discussion, strategies of questioning, testing, and either proving or disproving, could be pursued visibly by the collective.

A fourth instructional move employed regularly by the teacher was to create frequent pauses for *reflection* on progress, questions, stuckness, and new insights. She provided supports for reflecting.

One such support was the use of sentence stems to prompt a specific kind of pause and reflection—e.g., “Before we stop for today, please write in your notebook, “I think we will be able to build Mr. X’s train because _____________,” or “I don’t think we will be able to build Mr. X’s train because _____________.” Some prompts were less structured—e.g., “Write one thing you figured out today.” Using prompts was a strategy for externalizing the sorts of questions to pose to oneself as a means of persevering with a hard problem.

Taking Stock and Next Steps

While psychologists investigate differences in people’s persistence in the face of difficulty, children and adults also exercise varying perseverance across the challenges they encounter. A three-year old girl may try over and over to build a very high and complex tower with blocks without it falling down, but when she tries to zip her coat becomes almost immediately frustrated. Her skills and techniques are different for these two tasks. We take the perspective that perseverance is a domain-specific bundle of skills that support confidence in the struggle to succeed. School mathematics is filled with moments of frustration and being stuck, but students too often are given encouragement, almost cheerleading (i.e., “You can do it!”), without learning the specific practices that enable successful perseverance with complex mathematics. To unpack this argument, we traced in this chapter the instructional sequence in a fifth grade class in which students worked together on a very challenging mathematics problem. We identified mathematical and contextual features of the task that set up an authentic engagement in a complex problem, the staging of the students’ work on the problem, the social structures of the work in the class, and the instructional approach and techniques used. The collective positioning of the mathematics problem helped to support making the work of persisting visible, and created opportunity for discussion and other ways of externalizing the work.

Ours was an exploration of what it might mean to do more than encourage students to persevere with complex mathematics. Important questions are raised by this that merit study and deeper investigation. One centers on our hypothesis that to be helped to persevere with a particular problem helps students to build more general strategies for mathematical perseverance. Because we conceive of perseverance as a bundle of practices, we posit that learning is best situated in actual cases where doing is entailed. If students are helped to work on complex and challenging problems, and succeed at them, does this also set them up to know what to do when they are working alone on another problem? If it does, to what sort of range of mathematical problems and contexts might such more general learning extend? How domain-specific is mathematical perseverance?

Other big questions have to do with the role of believing in one’s own ability to solve a problem and of motivation. We are arguing that a critical component of perseverance is skill and technique. But does confidence play a role, and if so, what sort of confidence and how does it interact with technique, if at all? Does caring about the goal matter, and if so how does this play out?

A third has to do with the measurement of mathematical perseverance. To study the sorts of questions identified above would require sensitive and valid measures of persistence with difficult mathematics. Developing reliable ways to do this would advance our understanding of efforts to develop perseverance, as well as extend conceptualization of perseverance itself.

Finally, we became interested in the possibility that perseverance may be a collective as well as an individual capability. Can perseverance be usefully construed as distributed among members of a group and if so, how might this be both cultivated and used?

Coda: Knowing When to Stop

We close with the observation that perseverance not only requires persistence and determination, but can also bring rewards of closure. When the students presented to Mr. X their report that the train he ordered was impossible, he displayed annoyance. He would order it from another train company that would try harder and work until the train was built, he told the students. But one of the students, fortified by the thoroughness of the class's persistence and their collective work, replied with assurance, "It doesn't matter how long they work or how hard, it is IMPOSSIBLE. We proved it." Their skill in persevering to the conclusion and the construction of the proof had produced a firm, albeit disappointing, mathematical result. They knew that they were done, and that further persistence would be fruitless.

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